Irregular Labelings of Circulant Graphs

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Notation

- $G$ - (connected) simple graph with no isolated edges and at most one isolated vertex
- $E(G)$ - the edge set of $G$
- $V(G)$ - the vertex set of $G$
- $n = |V(G)|$
- $d_G(v)$ - the degree of vertex $v$ in $G$
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- $d_G(v)$ - the degree of vertex $v$ in $G$
Assign positive integer $w(e) \leq s$ to every edge $e \in E(G)$.

- For every vertex $v \in V(G)$ the weighted degree is defined as:

$$wd(v) = \sum_{e \ni v} w(e).$$

- $w$ is irregular if for $v \neq u$ we have $wd(v) \neq wd(u)$.

- *Irregularity strength* $s(G)$: the lowest $s$ that allows some irregular labeling.
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\( s(G) \): Some results

- Lower bound:

\[
s(G) \geq \max_{1 \leq i \leq \Delta} \frac{n_i + i - 1}{i}
\]

- Best upper bound (M. Kalkowski, M. Karoński, F. Pfender, 2009):

\[
s(G) \leq \left\lceil \frac{6n}{\delta} \right\rceil
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- Exact values for some families of graphs (e.g. cycles, grids, some kinds of trees).
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**TVS(G): Definition**

Assign positive integer \( w(e) \leq s \) to every edge \( e \in E(G) \) and every vertex \( v \in V(G) \).

- For every vertex \( v \in V(G) \) the *weighted degree* is defined as:
  \[
  wd(v) = w(v) + \sum_{e \ni v} w(e).
  \]

- \( w \) is irregular if for \( v \neq u \) we have \( wd(v) \neq wd(u) \).

- *Total vertex irregularity strength TVS(G):* the lowest \( s \) that allows some irregular labeling.
**tv(s(G)): Definition**

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**tvs(G):** Definition

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$tvs(G)$: Some results

- Lower bound:

$$tvs(G) \geq \left\lceil \frac{n + \delta(G)}{\Delta(G) + 1} \right\rceil$$

- Best upper bound (M. Anholcer, M. Kalkowski, J. Przybyło, 2009):

$$tvs(G) \leq \left\lceil \frac{3n}{\delta} \right\rceil + 1.$$  

- Exact values for some families of graphs (e.g. cycles, prisms, some kinds of forests).
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- Exact values for some families of graphs (e.g. cycles, prisms, some kinds of forests).
**tvS(G): Some results**

- Lower bound:

\[ tvS(G) \geq \left\lceil \frac{n + \delta(G)}{\Delta(G)} + 1 \right\rceil \]

- Best upper bound (M. Anholcer, M. Kalkowski, J. Przybyło, 2009):

\[ tvS(G) \leq \left\lceil \frac{3n}{\delta} \right\rceil + 1. \]

- Exact values for some families of graphs (e.g. cycles, prisms, some kinds of forests).
Circulant Graphs

Definition

Let $n$ and $s_1, s_2, \ldots, s_k$ be integers, with $1 < s_1 < \cdots < s_k \leq n/2$. The circulant graph $G = C_{i_{n}}(s_1, \ldots, s_k)$ of order $n$ is a graph with vertex set $V(G) = \{0, 1, \ldots, n-1\}$ and edge set $E(G) = \{(x, x \pm s_i \mod n), x \in V(G), 1 \leq i \leq k\}$. 
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Theorem (J.-L. Baril, H. Kheddouci, O. Togni, 2005)

If $k = 2$ and $s_1 = 1$, then

$$s(C_{i_n}(1, s_2)) = \left\lceil \frac{n + 3}{4} \right\rceil.$$
Main Results

Theorem (MA, 2010)

If \( k \geq 2 \) and \( n \geq 2k + 1 \) then

\[ \text{tv}(Ci_n(1, 2, \ldots, k)) = \left\lceil \frac{n + 2k}{2k + 1} \right\rceil. \]

Theorem (MA, 2010)

If \( k \geq 2 \) and \( n \geq 2k + 1 \) then

\[ s(Ci_n(1, 2, \ldots, k)) = \begin{cases} 
\left\lceil \frac{n + 2k - 1}{2k} \right\rceil + 1 & n = (4t + 2)k + 1, \ k \text{ odd or } t = 0, \\
\left\lceil \frac{n + 2k}{2k} \right\rceil - 1 & \text{otherwise.}
\end{cases} \]
Lemma 1

![Graph Diagram]

0 2 4 6 8
0 0 1 2 1 2 1
(l = 2) (l = 4) (l = 0) (l = 0) (l = 0)
Lemma 2

\[ l = 0 \]

\[ l = 0 \]

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\[ l = 2 \]

\[ l = 4 \]
Lemma 3

```
2 3 4 1 0
1 1 1 0
1 1 1 -1
```

```
3 4 5 2 1
1 1 1 0
1 1 1 -1
```

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Main Results

Remark

\[ Ci_n(s_1, \ldots, s_k) = C_n^k \]

Theorem (MA, 2010)

If \( k \geq 2 \) and \( n \geq 2k + 1 \) then

\[ tvs(Ci_n(1, 2, \ldots, k)) = \left\lceil \frac{n + 2k}{2k + 1} \right\rceil. \]

Theorem (MA, 2010)

If \( k \geq 2 \) and \( n \geq 2k + 1 \) then

\[ s(Ci_n(1, 2, \ldots, k)) = \begin{cases} 
\left\lceil \frac{n+2k-1}{n+2k-1} \right\rceil + 1 & n = (4t + 2)k + 1, \; k \text{ odd or } t = 0, \\
\text{otherwise.} & \end{cases} \]
Case 1: \( k \in \left\{ \frac{n-2}{2}, \frac{n-1}{2} \right\} \).

In such a situation \( Ci_n(1, 2, \ldots, k) = K_{2k} \) or \( Ci_n(1, 2, \ldots, k) = K_{2k+1} - M \) and we use special labeling.
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In such a situation $Ci_n(1, 2, \ldots, k) = K_{2k}$ or $Ci_n(1, 2, \ldots, k) = K_{2k+1} - M$ and we use special labeling.
Case 2: \( k \leq \frac{n-3}{2} \) (equivalent to \( n \geq 2k + 3 \)).

Then \( n = t(4k + 2) + r \), where \( t \geq 0 \) and \( 1 \leq r \leq 4k + 2 \).

Let \( s = \left\lceil \frac{n+2k}{2k+1} \right\rceil \) (Thm 1) or respectively \( s = \left\lceil \frac{n+2k-1}{2k} \right\rceil \) (Thm 2).
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Sketch of the proofs

\[
\begin{align*}
\text{(l=0)} & & \text{(l=0)} & & \text{(l=4)} & & \text{(l=4)} & & \text{(l=2)} & & \text{(l=0)} & & \text{(l=0)} & & \text{(l=0)} \\
0 & & 2 & & 4 & & 6 & & 8 & & 8 & & 6 & & 4 & & 2 & & 0 \\
(0) & & (0) & & (0) & & (1) & & (2) & & (4) & & (2) & & (2) & & (1) & & (0) \\
0 & & 2 & & 4 & & 6 & & 8 & & 8 & & 6 & & 4 & & 2 & & 0 \\
(0) & & (0) & & (0) & & (0) & & (0) & & (1) & & (0) & & (0) & & (0) & & (1) & & (2) \\
0 & & 2 & & 4 & & 6 & & 8 & & 8 & & 6 & & 4 & & 2 & & 0 \\
(0) & & (0) & & (0) & & (0) & & (0) & & (0) & & (1) & & (0) & & (0) & & (1) & & (2) & & (4) \\
\end{align*}
\]
Multiply edge labels by $\frac{s-1}{2}$, then add 1 and modify if $s$ is even.

- Not more than $s - 1$ occurrences of every weighted degree.
- Weighted degrees differ by at least $s - 1$.
- In order to distinguish add vertex labels or reduce edge labels through factors $F$. 

![Diagram of graph with labeled vertices and edges]
Sketch of the proofs

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