VRP with Time Windows and Other Time Constraints – Modelling and Solution Approaches¹

Marcin Anholcer*, Arkadiusz Kawa**

*Department of Operations Research, Poznań University of Economics and Business, Al. Niepodległości 10, 61-875 Poznań, Poland (Tel: +48 61 856 9545; e-mail: m.anholcer@ue.poznan.pl).

** Department of Logistics and Transport, Poznań University of Economics and Business, Al. Niepodległości 10, 61-875 Poznań, Poland (Tel: +48 61 854 3463; e-mail: arkadiusz.kawa@ue.poznan.pl).

Abstract: In this paper we study the Vehicle Routing Problem with time windows and other possible exclusions concerning time. After a brief introduction we present the problem of exclusionary constraints, in particular the ones connected with time. Then we present the mathematical model of the problem. Finally we discuss possible solution strategies.

Keywords: Exclusionary constraints, transportation, vehicle routing, time windows.

1. INTRODUCTION

Supply chain management (SCM) is a sequence of events in a goods flow within an inter-organizational network. In this flow the value is added to a specific good. The main objective of the supply chain is to provide the maximum value to customers at low costs and high speed (Chen, Daugherty and Roath 2009).

The literature on the subject and economic practice distinguish at least several types of supply chains depending mainly on the characteristics of the product or industry

What is closely connected with the flow of goods among the entities of an inter-organizational network is transport. Transport is one of the key processes in a supply chain. It enables spatial configuration of separate organizations, which exchange goods between one another. It constitutes a significant part of logistics expenses, so it requires appropriate planning and effective control. The properties and possibilities of means of transport should be taken into account, just as many assumptions concerning transport organization, such as loading and unloading places, transport time and load weight.

In case of many sectors of the economy (e.g. food industry), the enterprises use various suppliers and sell their products to many buyers. This causes problems with delivery coordination, which affects the quality and costs of products. The lack of planning and making wrong decisions may cause excessive expenses, an inability to achieve the assumed goals and a loss of any chances to get a profit.

The key issue in transport are, then, its costs. They can be a significant part of a company’s overall logistics spending (Murray 2016). According to various estimates, transportation constitutes one-third of the logistics costs (Tseng, Yue and Taylor 2005; Ślusarczyk and Kot 2013, Michałowska 2013). All the costs are transferred to the customer who needs to pay more for the goods. That is why cutting transportation costs is the major target for companies. There is a number of strategies that can be applied by management to help improve performance. Only some of them use complex mathematical methods enabling cost optimization. The issue of transportation cost optimization is, however, treated as one of the most difficult and, at the same time, most complex problems which transport enterprises deal with (Romanow 2013).

For that reason, in this paper we discuss one of the problems connected with the transportation of goods. To be more specific, we focus on the problems with various kinds of additional constraints connected with time.

2. EXCLUSIONARY CONSTRAINTS

2.1 Exclusionary constraints in transportation

Exclusionary (or incompatibility) constraints in transport are the circumstances or conditions under which at least two deliveries cannot be made at the same time, place or by the same mean of transport.

One may distinguish several common reasons that may cause the exclusions. The most frequent are:

-- sensitivity for the time of delivery,
-- sensitivity for transportation conditions (eg. items that are fragile, prone to damage),
-- sensitivity for temperature,
-- sensitivity for humidity,

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- sensitivity for light,
- sensitivity for smells,
- perishability of chosen products,
- oversize of chosen products,
- competition,
- time windows,
- specific agreements,
- long distance from the loading or unloading sites,
- legal prohibitions.

All of those constraints make the problems to solve much more complicated – in particular the classical network flow algorithms may not be applied.

2.2 Time exclusions in transportation

In this paper we focus in particular on the exclusions connected with time windows. To be more specific, we are interested not only in the fixed time windows, but also in the exclusionary constraints caused by the restrictions on performing two operations at the same time.

Our previous research (Anholcer and Kawa 2017), conducted among Polish companies dealing with logistics, shows that the time exclusions are both frequent and significant for many of them. In particular, almost 20% of respondents declared that time exclusions occur very often, while over 30% of them recognized them as significant for the company (see Figures 1 and 2).

3. VEHICLE ROUTING PROBLEMS WITH TIME CONSTRAINTS

One of the most important kinds of the transportation problems is to optimally plan the routes of trucks delivering the goods from the depot to customers. In the case of one truck, this is the Traveling Salesman Problem (TSP), while in the case of many trucks – the Vehicle Routing Problem (VRP), both well-known and having been studied in numerous papers (for a comprehensive review, see e.g. (Punnen 2004) and following parts of the same book).

TSP as well as VRP with time constraints have been also extensively studied, see e.g. (Baker 1983, Desrosiers 1988, Kolen et al. 1987, Savelsbergh 1985, Solomon 1987).

What is important from our point of view, the VRP with time windows can be modelled as a Minimum Cost Flow Problem (see Ahuja et al. 1993) if one uses the approach presented e.g. by Schneider (2016). As we will show in the next section, also other time exclusions can be modelled this way.

4. MATHEMATICAL MODEL

4.1 Case 1: collisions cannot occur

For the beginning we assume, that there are some time windows within which the deliveries need to take place, however there are no other time restrictions. In particular, we assume that the collisions are impossible (by collision we do not mean an accident, which can always occur, but a situation where one vehicle blocks another, so that the latter one cannot perform the planned operations).

We define the problem as usual, i.e. using the network model, where the nodes represent the pairs destination–order (so that we consider also the situation where there may be several orders from same destination), or simply order (for every order, the destination is uniquely defined). In case of standard VRP with single orders, the nodes are simply the destinations.
The mathematical model that we are going to present bases on the model for VRP with time windows that one can find e.g. in (Schneider 2016). We assume identical vehicles with capacity $C$ delivering uniform commodity to several destinations. The set of vehicles is indexed by $1, ..., K$.

Every order is represented by a node $v_1, v_2, ..., v_n$, where $n$ is the number of orders. The warehouse is represented by two extra nodes – source and sink, denoted respectively by $v_0$ and $v_{n+1}$. For every node $v_i$ the following parameters are defined:

- $s_i$ – service time of order $i$,
- $e_i$ – earliest moment of processing order $i$,
- $l_i$ – latest moment of processing order $i$,
- $q_i$ – size (demand) of order $i$.

For every pair of nodes ($v_i, v_j$), the transportation time $t_{ij}$ is known. Following variables have been used:

- $x_{ijk}$ – binary variable equal to 1 if and only if vehicle $k$ processes order represented by node $v_j$ directly after the order represented by $v_i$.
- $y_{ik}$ – continuous variable representing the starting moment of processing order at $v_j$ by vehicle $k$.

We want to minimize the total processing time, so the objective is as follows:

$$\min (y_{n+1,k} - y_{0,k}).$$

Let us describe the constraints. Each order must be processed:

$$\sum_{k=1}^{K} \sum_{j=1}^{n+1} x_{ijk} = 1, i = 1, ..., n.$$

Every route starts in the warehouse:

$$\sum_{j=1}^{n+1} x_{0jk} = 1, k = 1, ..., K.$$

Mass balance constraints must be satisfied:

$$\sum_{i=0}^{n} x_{ijk} - \sum_{i=1}^{n+1} x_{jik} = 0, j = 1, ..., n, k = 1, ..., K.$$

The time between starting two consecutive deliveries cannot be shorter than the sum of service time and transportation time ($M$ is a large constant):

$$y_{jk} - y_{ik} + M (1 - x_{ijk}) \geq t_{ij} + s_i, i = 0, ..., n + 1, j = 0, ..., n + 1, i \neq j, k = 1, ..., K.$$

Delivery must take place inside the time window:

$$e_i \leq y_{ik} \leq l_i, i = 0, ..., n + 1, k = 1, ..., K.$$

Total amount of delivered goods cannot exceed the vehicle’s capacity:

$$\sum_{i,j=1,...,n+1, i \neq j} q_i x_{ijk} \leq C, k = 1, ..., K.$$

Common time window for two extra nodes is defined as:

$$e_0 = e_{n+1} = \min\{e_i | i = 1, ..., n\},$$

$$l_0 = l_{n+1} = \max\{l_i | i = 1, ..., n\}.$$

Binarization of selected variables:

$$x_{ijk} \in \{0,1\}, i = 0, ..., n + 1, j = 0, ..., n + 1, i \neq j, k = 1, ..., K.$$

4.2 Case 2: collisions can occur and must be excluded by the model

Sometimes the collisions may occur. We can distinguish two kinds of them.

First possibility is when two vehicles reach the same section of a narrow road at the same time. This kind of situations is typical for the automated deliveries in factories, e.g. Milk Run systems (see e.g. Pawlewski and Anholcer 2018). Another problem, that can occur in any setting, is the meeting of two vehicles at the same time at one destination. In order to avoid such situations, we are going to introduce new variables, representing the end times of deliveries:

$$z_{ik}$$

continuous variable representing the ending moment of processing order at $v_j$ by vehicle $k$.

The mathematical model looks now as follows. The objective and several first constraints are exactly the same or very similar to the ones from the original model:

$$\min (y_{n+1,k} - y_{0,k}).$$

$$\sum_{k=1}^{K} \sum_{j=1}^{n+1} x_{ijk} = 1, i = 1, ..., n,$$

$$\sum_{j=1}^{n+1} x_{0jk} = 1, k = 1, ..., K,$$

$$\sum_{i=0}^{n} x_{ijk} - \sum_{i=1}^{n+1} x_{jik} = 0, j = 1, ..., n, k = 1, ..., K,$$

The new constraint appears – the difference between starting and ending the service cannot be smaller than the service time:

$$z_{ik} - y_{ik} \geq s_i, i = 0, ..., n + 1, k = 1, ..., K.$$

Time between ending an order and starting next one cannot be shorter than the transportation time:

$$y_{jk} - z_{ik} + M (1 - x_{ijk}) \geq t_{ij} + s_i, i = 0, ..., n + 1, j = 0, ..., n + 1, i \neq j, k = 1, ..., K.$$

Time window constraints and capacity constraints do not change:

$$e_i \leq y_{ik} \leq l_i, i = 0, ..., n + 1, k = 1, ..., K,$$

$$\sum_{i,j=1,...,n+1, i \neq j} q_i x_{ijk} \leq C, k = 1, ..., K,$$
\[
e_0 = e_{n+1} = \min \{ e_i \mid i = 1, ..., n \},
\]
\[
l_0 = l_{n+1} = \max \{ l_i \mid i = 1, ..., n \}.
\]

Same with the binarity constraint:
\[
x_{ijk} \in \{0, 1\}, i = 0, ..., n + 1, j = 0, ..., n + 1, i \neq j, k = 1, ..., K.
\]

Now we are going to present the constraints that allow to avoid the collisions. For vehicle \(k\), the period of traveling between nodes \(v_i\) and \(v_j\) is represented by the interval
\[
\left[ z_{ijk}, y_{jk} \right], \quad \text{if } x_{ijk} = 1,
\]
\[
\emptyset, \quad \text{if } x_{ijk} = 0.
\]

In order to avoid collision, the following constraint must be satisfied for every pair of orders \((i, j)\) and every pair of trains \((k, k')\):
\[
\left( (x_{ijk} = x_{ijk'} = 1) \land z_{ik} \geq z_{ik'} \right) \Rightarrow y_{jk} \geq y_{jk'}, i, j
\]
\[
= 0, ..., n + 1, i \neq j, k, k' = 1, ..., K, k \neq k'.
\]

In fact, one can use tight inequalities, since the possibility of the collision in the beginning or end of any connection between nodes will be excluded by the constraints forbidding collisions at nodes (see below):
\[
\left( (x_{ijk} = x_{ijk'} = 1) \land z_{ik} \geq z_{ik'} \right) \Rightarrow y_{jk} \geq y_{jk'}, i, j
\]
\[
= 0, ..., n + 1, i \neq j, k, k' = 1, ..., K, k \neq k'.
\]

If two-way connections do not allow two vehicles to pass by, the following constraint must be included in the model:
\[
\left( x_{ijk} = x_{ijk'} = 1 \right) \Rightarrow (y_{jk} < z_{jk} \lor z_{ik} \geq y_{jk'}), i, j
\]
\[
= 0, ..., n + 1, i \neq j, k, k' = 1, ..., K, k \neq k'.
\]

Also in this case it can be transformed:
\[
\left( x_{ijk} = x_{ijk'} = 1 \right) \Rightarrow (y_{jk} \leq z_{jk} \lor z_{ik} \geq y_{jk'}), i, j
\]
\[
= 0, ..., n + 1, i \neq j, k, k' = 1, ..., K, k \neq k'.
\]

Another kind of collision may occur, when two vehicles are at the same time at the same node. The period spent at node \(v_i\) by vehicle \(k\) is represented by the interval:
\[
[y_{ik}, z_{ik}].
\]

For every two vehicles \(k\) and \(k'\) and every node \(i\) the intervals \([y_{ik}, z_{ik}]\) and \([y_{ik'}, z_{ik'}]\) must be disjoint:
\[
\left( \exists j = 0, ..., n + 1 \right) (x_{ijk} = 1) \land \exists j = 0, ..., n + 1 \right) x_{ijk'} = 1
\]
\[
\Rightarrow (y_{ik} > z_{ik'} \lor z_{ik} < y_{ik'}),
\]
\[
i = 0, ..., n + 1, i \neq j, k, k' = 1, ..., K, k \neq k'.
\]

In order to use tight inequalities, one needs to introduce a very small positive constant \(\varepsilon\):
\[
\left( \exists j = 0, ..., n + 1 \right) (x_{ijk} = 1) \land \exists j = 0, ..., n + 1 \right) x_{ijk'} = 1
\]
\[
\Rightarrow (y_{ik} \geq z_{ik'} + \varepsilon \lor z_{ik} \leq y_{ik'} - \varepsilon),
\]
\[
i = 0, ..., n + 1, i \neq j, k, k' = 1, ..., K, k \neq k'.
\]

One can also use the following, equivalent form:
\[
\left( \sum_{j=0, n+1} x_{ijk} = 1 \land \sum_{j=0, n+1} x_{ijk'} = 1 \right)
\]
\[
\Rightarrow (y_{ik} \geq z_{ik'} + \varepsilon \lor z_{ik} \leq y_{ik'} - \varepsilon),
\]
\[
i = 0, ..., n + 1, k, k' = 1, ..., K, K \neq k'.
\]

The constraints presented above have specific form of alternatives or implications. If one wants to solve the problem directly, they need to use a Constraint Programming solver. Another approach is to introduce new binary variables and transform the logical constraints to algebraic ones. We are going to discuss the second approach in the remainder of this section.

Assume that we want to transform the constraint:
\[
\left( (x_{ijk} = x_{ijk'} = 1) \land z_{ik} > z_{ik'} \right) \Rightarrow y_{jk} > y_{jk'}.
\]

We start with elimination of implications:
\[
x_{ijk} = 0 \lor x_{ijk'} = 0 \lor z_{ik} \leq z_{ik'} \lor y_{jk} > y_{jk'}.
\]

We introduce switch variables \(a^1_{ijk}\) and \(a^2_{ijk}\) for the third and fourth expression:
\[
a^1_{ijkk} = \begin{cases} 1, & \text{if } z_{ik} \leq z_{ik'}, \\ 0, & \text{if } z_{ik} > z_{ik'}. \end{cases}
\]
\[
a^2_{ijkk} = \begin{cases} 1, & \text{if } y_{jk} > y_{jk'}, \\ 0, & \text{if } y_{jk} \leq y_{jk'}. \end{cases}
\]

In order to guarantee the connection, we use the constraints:
\[
z_{ik} \leq z_{ik'} + (1 - a^1_{ijkk}) M,
\]
\[
z_{ik} \geq z_{ik'} - a^2_{ijkk} M.
\]

Indeed, if \(a^1_{ijk} = 1\), then we have:
\[
z_{ik} \leq z_{ik'},
\]
\[
z_{ik} \geq z_{ik'} - M.
\]

Similarly, if \(a^1_{ijk} = 0\), then we have:
\[
z_{ik} \leq z_{ik'} + M,
\]
\[
z_{ik} \geq z_{ik'}.
\]

We define the constraints for the fourth expression in analogous way. Now, if we remember that the alternative of four expressions is true if and only if at least one of them is true, the initial condition
\[
\left( (x_{ijk} = x_{ijk'} = 1) \land z_{ik} > z_{ik'} \right) \Rightarrow y_{jk} > y_{jk'}
\]

can be rewritten as the following system of linear constraints:
\[
z_{ik} \leq z_{ik'} + (1 - a^1_{ijkk}) M,
\]
\[
z_{ik} \geq z_{ik'} - a^1_{ijkk} M,
\]
\[ y_{jk} \leq y_{jk'} + \alpha_{ikj}^2 M, \]
\[ y_{jk} \geq y_{jk'} - (1 - \alpha_{ikj}^2) M, \]
\[ (1 - x_{ijk}) + (1 - x_{ijk'}') + \alpha_{ikj}^1 + \alpha_{ikj}^2 M \geq 1, \]
\[ a_{ijk'}', a_{ikj}^2 \in [0,1]. \]

In the case of collisions at nodes, the respective constraint had the form:
\[ (\exists j = 0, \ldots, n+1 x_{ijk} = 1 \land \exists j = 0, \ldots, n+1 x_{ijk'} = 1) \Rightarrow (y_{ik} > z_{ik'} \lor z_{ik} < y_{ik'}'). \]

Using similar reasoning, one can transform it to the following linear system:
\[ y_{ik} \geq z_{ik'} - (1 - \beta_{ikj}^1) M + \varepsilon, \]
\[ y_{ik} \leq z_{ik'} + \beta_{ikj}^1 M, \]
\[ z_{ik} \leq y_{ik'} + (1 - \beta_{ikj}^2) M - \varepsilon, \]
\[ z_{ik} \geq y_{ik'} - \beta_{ikj}^2 M, \]
\[ (1 - \sum_{j=0, \ldots, n+1} x_{ijk}) + (1 - \sum_{j=0, \ldots, n+1} x_{ijk'}') + \beta_{ikj}^1 \]
\[ + \beta_{ikj}^2 M \geq 1, \]
\[ \beta_{ikj}^1, \beta_{ikj}^2 \in [0,1]. \]

By repeating the same procedure for every exclusionary constraint, one can transform the problem with logical constraints to the form of MILP (Mixed Integer Linear Programming) Problem, where the integer variables are in fact binary.

5. SOLUTION STRATEGIES

There are several possible solution strategies for the problems discussed in this paper.

If we consider the form with logical constraints, one can find an exact solution using one of the Constraint Programming solvers. They usually exploit the branch and bound framework to solve the problem.

If one transforms the logical constraints to the form of linear systems (as described in the preceding section), then any MILP solver can be used. What is important, in such case not only branch and bound but also faster algorithms can be used, like branch and cut or branch and price. Of course the problem transformed to the form of MILP is much larger, so a faster (in general) algorithm may need more time to solve it.

Unfortunately, no matter which approach would be chosen, if the problem is large enough (what very likely may happen in practice), the solution time would be unacceptable. This means that a heuristic approach should be chosen.

There are many heuristics that may be used. However we suggest the approach described by Pawlewski and Anholcer (2018). A two-phase method was presented there. Let us briefly present it here.

In the first phase the orders are assigned to vehicles by a heuristic that guarantees that two conditions are satisfied: the capacity constraint of any vehicle will not be violated and every order will be serviced in the respective time window. Checking the first condition is trivial, while in order to make sure that the second one is satisfied, one needs to solve the variant of the TSP presented below (we present the version where we assume that collisions are impossible, the auxiliary problem for the version with additional constraints can be developed in an analogous way).

The variables are:
\[ x_{ij} - \text{binary variable equal to 1 if and only if the vehicle visits node } v_i \text{ directly after node } v_j, \]
\[ y_i - \text{continuous variable representing the starting moment of servicing the order at node } v_i. \]

The model takes the form:
\[ \min(y_{n+1} - y_0), \]
\[ \sum_{j=1}^{n+1} x_{ij} = 1, i = 1, \ldots, n, \]
\[ \sum_{j=1}^{n+1} x_{ij} = 1, \]
\[ \sum_{i=0}^{n} x_{ij} - \sum_{i=1}^{n+1} x_{ij} = 0, j = 1, \ldots, n, \]
\[ y_j - y_i + M (1 - x_{ij}) \geq t_{ij} + s_i, i = 0, \ldots, n + 1, j = 0, \ldots, n + 1, i \neq j, \]
\[ e_i \leq y_i \leq l_i, i = 0, \ldots, n + 1, \]
\[ \sum_{i=1}^{n+1} x_{ij} \leq C, \]
\[ e_0 = e_{n+1} = \min\{e_i | i = 1, \ldots, n\}, \]
\[ l_0 = l_{n+1} = \max\{l_i | i = 1, \ldots, n\}, \]
\[ x_{ij} \in \{0,1\}, i = 0, \ldots, n + 1, j = 0, \ldots, n + 1, i \neq j. \]

It is again a MILP problem, but much smaller. This means that it is more likely that it could be solved by some exact method in acceptable time. If it is still impossible, one can use some approximate methods, see e.g. (Syslo, Deo and Kowalik 1983) for reference.

6. CONCLUSIONS

The time exclusions make decision problems much more difficult to solve. It is also the case of transportation problems, including the Vehicle Routing Problems.

In order to find a solution in acceptable time, in practice it is very often necessary to use heuristic approach instead of exact
methods (like e.g. branch and bound, branch and cut or branch and price).

For that reason we suggest to further develop the heuristic algorithms for this kind of problems.

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